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13. ABSTRACT

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Piecewise Convex Activity Cost Functions						
Minimum Cost Project Schedule						
Convex Hull						
ATTACHMENT III (continued)						

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Technical Report #61

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by

Christian C. Robieux and Robert L. Sielken Jr.

Texas A&M University
Office of Naval Research
Contract N00014-78-C-0426
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ATTACHMENT I

PROJECT SCHEDULING WITH DISCONTINUOUS

PIECEWISE CONVEX ACTIVITY COST FUNCTIONS

by

Christian C. Robieux and Robert L. Sielken Jr.

THEMIS OPTIMIZATION RESEARCH PROGRAM
Technical Report No. 61
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ATTACHMENT II

ABSTRACT

When an activity can be performed with different techniques, the activity cost function may be a discontinuous piecewise linear or piecewise convex function of the activity's duration. This makes the determination of the minimum cost schedule satisfying a specified project deadline a nonconvex problem. However, if an activity may be performed using a combination of the different techniques, the concept of a convex hull can be used to transform the activity's cost function. The resulting convex problem can be solved by the existing PERT procedures. Therefore, this paper extends the applicability of existing PERT procedures to problems with discontinuous piecewise linear or piecewise convex activity cost functions.

PROJECT SCHEDULING WITH DISCONTINUOUS PIECEWISE CONVEX ACTIVITY COST FUNCTIONS

1. Introduction

The project scheduling procedure known as deterministic PERT solves the following problem. Let d_j be the deterministic duration for the j-th activity in a project network, and let $g_j(d_j)$ be the cost of completing activity j in d_j units of time. Let $D_P = \sum_{j \varepsilon P} d_j$ be the sum of the completion times on a path P linking the source with the sink. Then the minimum cost scheduling problem can be formulated as

$$\begin{aligned} &\text{minimize} \quad \sum_j g_j(d_j) \;, \\ &\text{subject to} \quad \max_P D_P \leq D \end{aligned} \tag{1}$$

where D is the specified completion time for the project. It is well-known that problem (1) can be formulated as a convex programming problem provided the $g_j(d_j)$ are convex functions. In case the $g_j(d_j)$ are linear or continuous piecewise linear the minimum cost project schedule can be efficiently determined using the algorithm described by Dunn and Sielken [1] which is based upon the earlier work of Fulkerson [2] and Lamberson and Hocking [3].

There are many activities where a convex cost function represents an oversimplification. In particular, we shall initially consider a discontinuous nonincreasing piecewise linear cost function. Such cost functions arise for instance if it is possible to complete an activity by a number of (say three) completely different techniques (Figure 1).

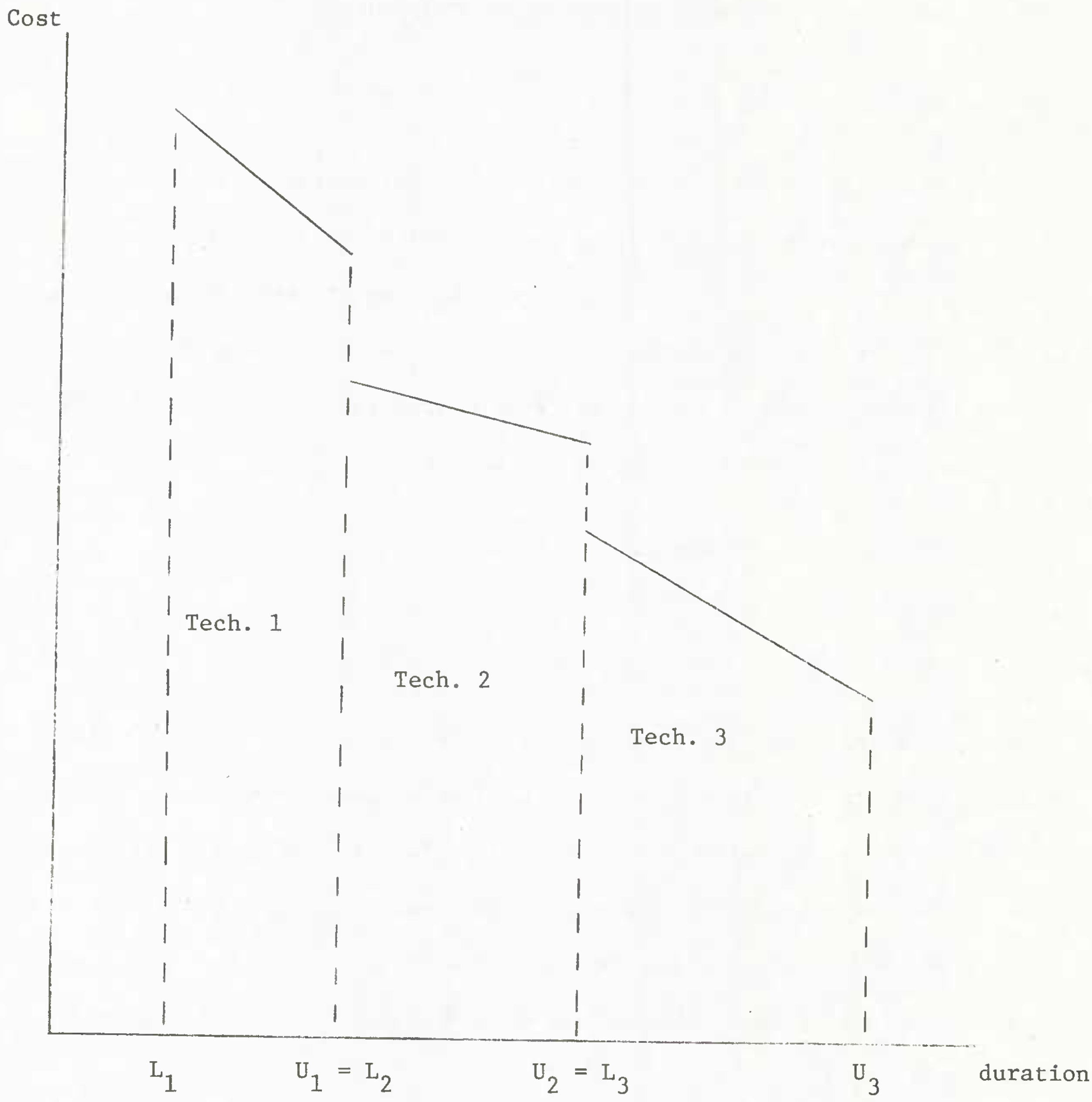

Figure 1

The Activity's Cost as a Function of its Duration

It is clear that the cost function plotted in Figure 1 is not a convex function and would lead to either nonconvex or integer programming, if we insist that the optimization should select one particular technique for the completion of the activity. However, there are many problems where it is much more reasonable to assume that a combination of techniques can be used to complete the activities. If a combination of techniques is used, it is quite possible that for the attainment of a particular duration t with say

$$L_2 \le t \le L_3$$

a combination of the three techniques may result in a cost that is lower than the ordinates given by the technique 2 cost line. This idea will be the basis of our approach.

For any particular activity, its duration, t, can be represented as

$$t = \sum_{i=1}^{n} p_i t_i$$

where there are $i = 1, \ldots, n$ techniques $(T_1, \ldots, T_n)$ with t_i representing the rate at which T_i is performed and p_i the fraction of the activity performed using T_i. The optimal mixture of techniques to perform an activity in a given duration t can be found by solving the following problem:

$$\text{minimize} \quad \sum_{i=1}^{n} p_i(\alpha_i + \beta_i t_i) \quad , \tag{2}$$

$$\text{subject to} \quad \sum_{i=1}^{n} p_i = 1 \quad , \tag{3}$$

$$\sum_{i=1}^{n} p_i t_i = t \quad , \tag{4}$$

$$L_i \le t_i \le U_i \ , \qquad i = 1, \ldots, n, \tag{5}$$

$$0 \le p_i \le 1 \ , \qquad i = 1, \ldots, n, \tag{6}$$

where the cost function for T_i has been written as $\alpha_i + \beta_i t_i$.

In Section 2 this nonlinear programming problem is transformed into an equivalent linear programming problem. Then, in Section 3, we shall show that in the linear programming problem the optimal objective function value (which equals the optimal activity cost) is a convex function of t. Hence any of the solution procedures that can be used to solve (1) when the activity costs, $g_j(t)$, are convex functions can also be used to solve (1) when the originally specified form of $g_j(t)$ is a discontinuous nonincreasing piecewise linear function. The establishment of this result is the primary purpose of this paper. An auxillary result noted in Section 3 is that the optimal mixture of techniques to perform the activity need not involve more than two techniques. Finally, in Section 4 the results for piecewise linear cost functions are extended to piecewise convex functions.

2. Linearizing the Nonlinear Problem

Theorem 1. The problem, P1, stated in (2) - (6), is equivalent to the problem, P2, given in (7) - (10):

$$\text{minimize} \quad \sum_{i=1}^{n} q_i(\alpha_i + \beta_i L_i) + q_{n+1}(\alpha_n + \beta_n U_n) \ , \tag{7}$$

$$\text{subject to} \quad \sum_{i=1}^{n+1} q_i = 1 \ , \tag{8}$$

$$q_i \ge 0 \ , \tag{9}$$

$$\sum_{i=1}^{n} q_i L_i + q_{n+1} U_n = t \ . \tag{10}$$

Proof: If $(p_1, \ldots, p_n; t_1, \ldots, t_n)$ is a feasible solution of P1, it can be shown that $(q_1, \ldots, q_{n+1}; L_1, \ldots, L_n, U_n)$ is a solution of P2 when

$$q_1 = p_1 r_1 \ ,$$

$$q_i = p_i r_i + (1 - r_{i-1}) p_{i-1} \ , \qquad i = 2, \ldots, n \ ,$$

$$q_{n+1} = p_n (1 - r_n) \ ,$$

where

$$r_i = \frac{L_{i+1} - t_i}{L_{i+1} - L_i} \ , \qquad i = 1, \ldots, n$$

and

$$L_{n+1} = U_n \ .$$

If C_1 and C_2 are the objective function values for these feasible solutions to P1 and P2 respectively, then using a little algebra

$$C_1 - C_2 = \sum_{i=1}^{n} p_i (1 - r_i) [\alpha_i + \beta_i L_{i+1} - (\alpha_{i+1} + \beta_{i+1} L_{i+1})] \ .$$

But the singularities of the activity's cost function are such that the jumps are positive when moving to shorter durations; i.e.

$$\alpha_i + \beta_i L_{i+1} > \alpha_{i+1} + \beta_{i+1} L_{i+1} \ , \qquad \text{for } i = 2, \ldots, n \tag{11}$$

Therefore, since $p_i \geq 0$ and $1 - r_i \geq 0$ for all i, (11) implies

$$C_1 - C_2 \geq 0 \ .$$

Thus, for any feasible solution of P1, there is a feasible solution of P2 with no greater cost. On the other hand, if $(q_1, \ldots, q_{n+1}; L_1, \ldots, L_n, U_n)$ is a feasible solution to P2, then

$$p_i = q_i \,, \qquad i = 1, \ldots, n-1$$

$$t_i = L_i \,, \qquad i = 1, \ldots, n-1$$

$$p_n = 1 - q_1 - \ldots - q_{n-1} \,,$$

$$t_n = \frac{q_n L_n + q_{n+1} L_{n+1}}{q_n + q_{n+1}} \,,$$

is a feasible solution to P1 with the same objective function value. Hence P1 and P2 are equivalent.

The importance of this result is that it is sufficient to study the problem P2 which is restricted to the convex hull, S, of the extremities of the segments on the activity's cost function graph.

3. Optimization on the Convex Hull

For a given activity duration, t, the lowest cost is attained by a point on the lower boundary of S as is evident in Figure 2. Since the lower boundary of a convex set is a convex function, the minimum activity cost is a convex function of t. Thus, if in the project scheduling problem in (1), $g_j(t)$ is given as a discontinuous piecewise linear nonincreasing function, it can be replaced by the convex function corresponding to the lower boundary of S (see for example Figure 3). This allows for (1) being solved by the existing deterministic project scheduling algorithms.

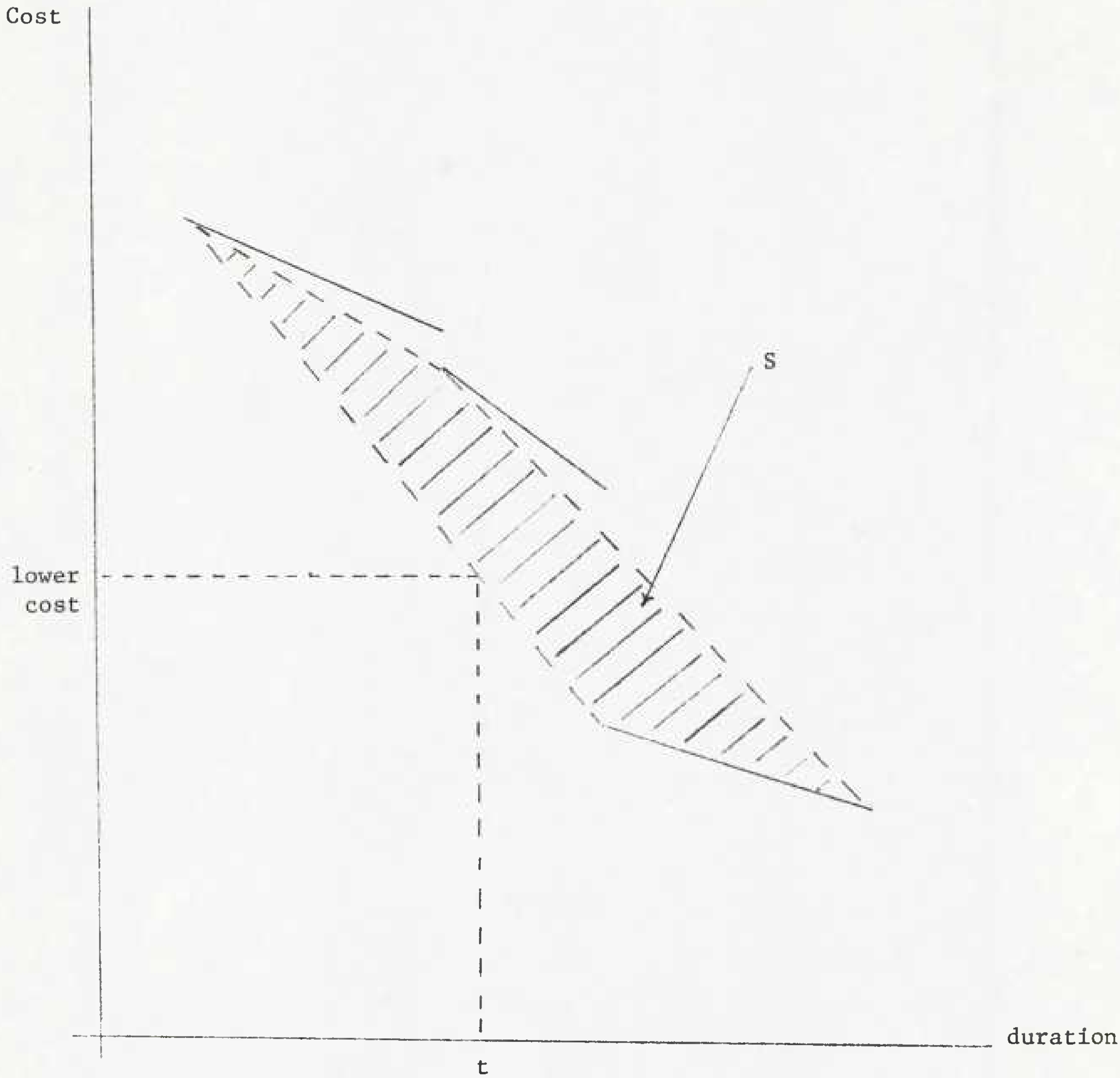

Figure 2

The Convex Hull of the Cost Function Graph

Extremities $L_1, \ldots, L_n, U_n$

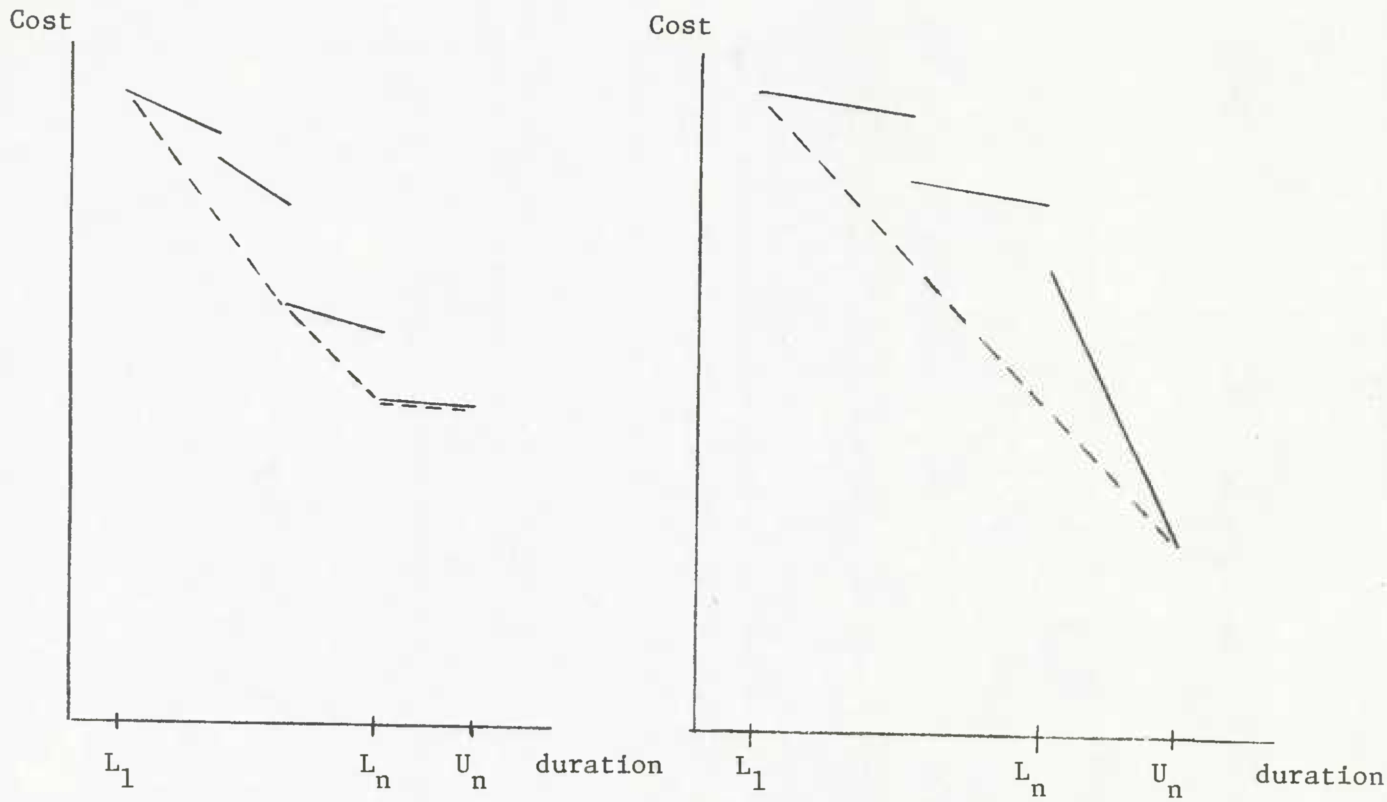

Figure 3

Lower Boundary of the Convex Hull

4. Generalization to a Discontinuous Nonincreasing Piecewise Convex Cost Function

For a given technique, it is more general to assume that the cost function is convex, see for example Figure 4.

The convex cost function of the piece corresponding to the technique T_i can be approximated by the continuous piecewise linear function that joins the points (t_k, cost (t_k)) for $k = 0, \ldots, m$ where

$$t_k = L_i + \frac{k}{m}(L_{i+1} - L_i) .$$

If we do this approximation for each technique T_i, we have as the whole range of time a discontinuous, nonincreasing, piecewise linear function. Then we are able to apply the theory of Sections 2 and 3. Since the convex hull of the graph of the approximate function tends for $m \to \infty$ to the convex hull of the graph of the initial piecewise convex cost function, the optimal strategy is to use the point on the lower boundary of the convex hull of the piecewise convex graph.

A point on this lower boundary is in one of two forms illustrated by M_1 and M_2 in Figure 5. The point M_1 is on the common tangent between two pieces of the cost function. The corresponding optimal mixture of techniques is to use the technique corresponding to A at the rate t_A to do $100p_A$% of the activity and the technique corresponding to B at the rate t_B to do the remaining $100(0 - p_A)$% of the activity where

$$M_1 = p_A A + (1 - p_A)B .$$

The savings realized by allowing a mixture of techniques is, in this case,

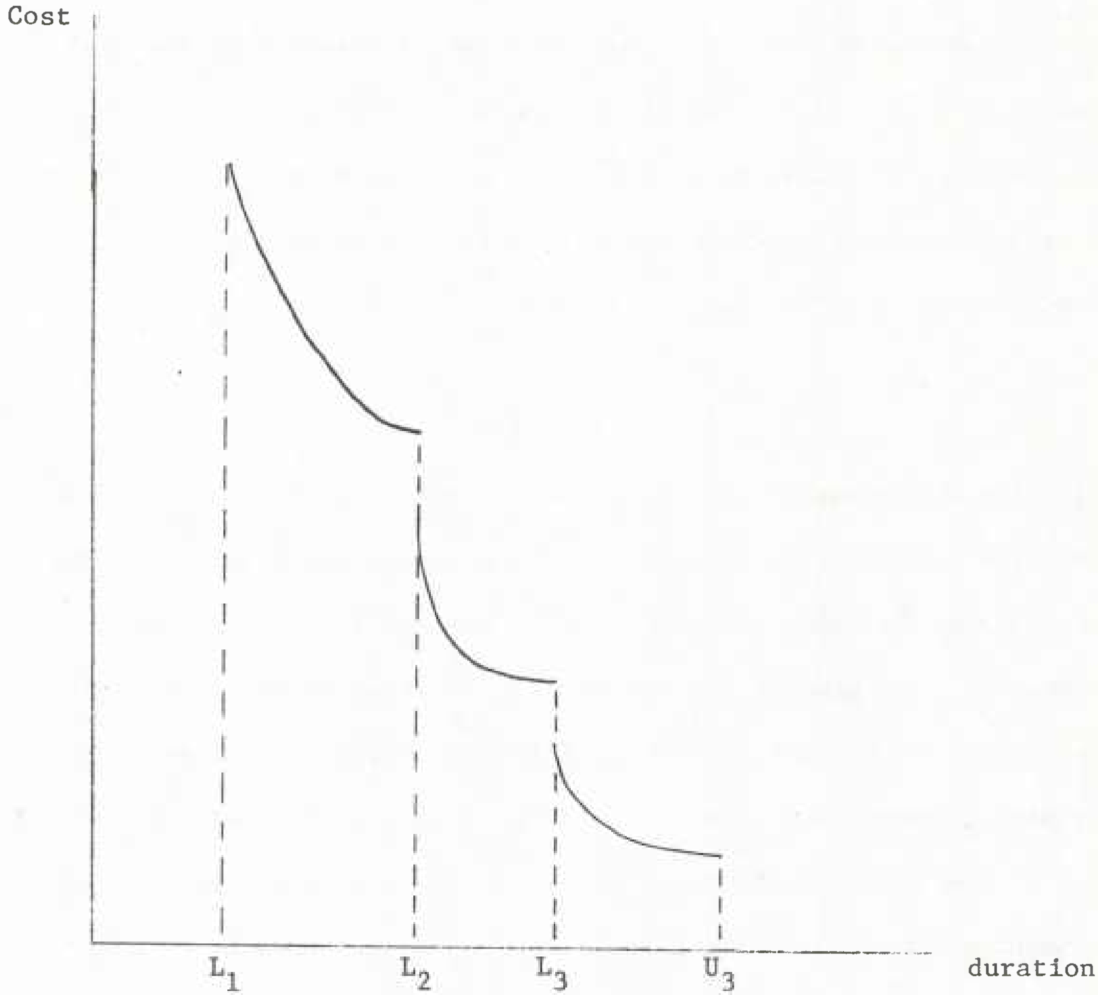

Figure 4

A Piecewise Convex Cost Function

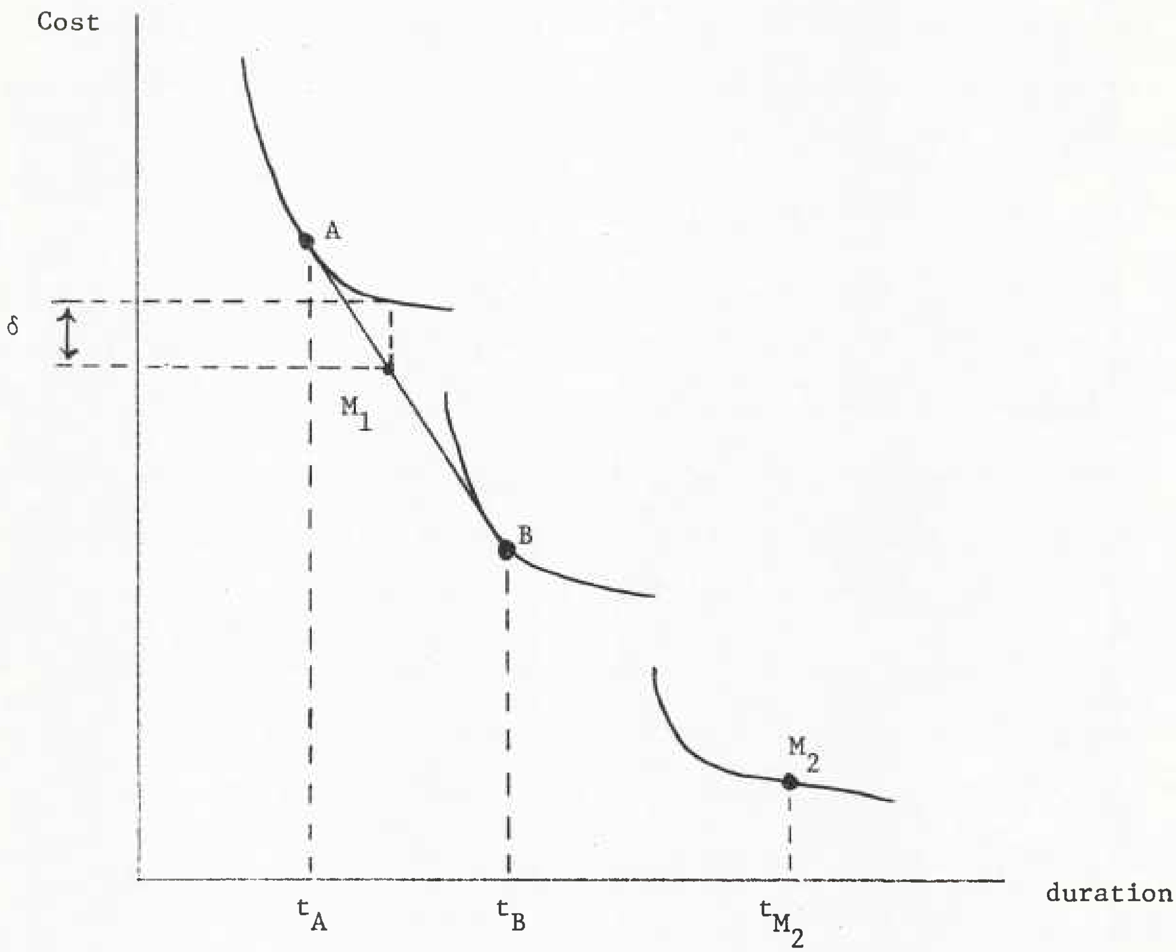

Figure 5

Combination of Techniques for a Piecewise Convex Cost Function

δ. On the other hand, the point M_2 is actually on the cost function for technique T_2, so the optimal strategy is simply to perform this activity using just T_2 at the rate t_{M_2}.

Once again, since the minimum cost for doing an activity corresponds to the lower boundary of a convex hull, the minimum activity cost is a convex cost function of t. Hence a specified discontinuous activity cost function $g_j(t)$ in (1) can be replaced by a convex function.

5. Conclusion

When an activity can be performed with different techniques, the activity cost function has a discontinuous piecewise nature. This makes project scheduling a nonconvex problem. However, if an activity may be performed using a combination of the different techniques, the concept of convex hull can be used to transform the activity's cost function. The resulting convex problem can be solved by the existing PERT procedures. Therefore, this paper extends the applicability of existing PERT procedures to problems with discontinuous piecewise linear or piecewise convex activity cost functions.

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